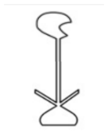
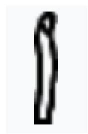
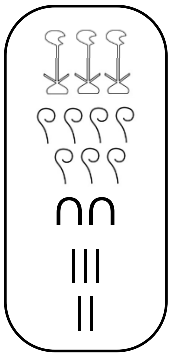
MATHS CLUB 27/11/25

EGYPTIAN FRACTIONS

CONCEPT: Ancient Egyptians (circa 3000 BC) counted base 10 like we do, and had symbols for 1, 10, 100 etc.

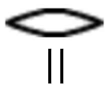
1	10	100	1 000	10 000	100 000	1 million or "many"
						

These were arranged vertically in patterns as additions such that

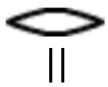
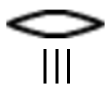


$= 3725$

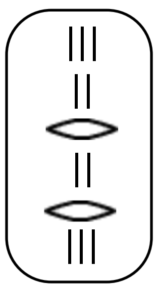
For fractions, ancient Egyptians used the mouth symbol  to mean "part".

 means "splitting into 2 parts". This is the ancient Egyptian for $\frac{1}{2}$.

Their only way to express fractions is **" one over … "**. We call this a **unit fraction**.

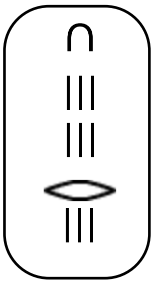
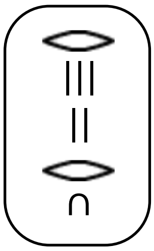
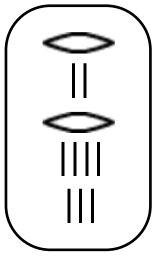
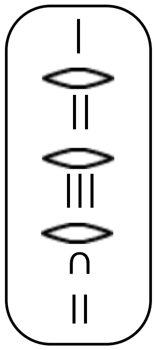
 $= \frac{1}{2}$	 $= \frac{1}{3}$	But there is no single symbol for $\frac{2}{3}$.

So all numbers need to be expressible as a sum of hundreds, tens, units etc. and unit fractions!
e.g.



$= 5 + \frac{1}{2} + \frac{1}{3} = \mathbf{5\frac{5}{6}} = \mathbf{\frac{35}{6}}$

TASK 1: Can you express the following hieroglyphs as fractions?

(a)	(b)	(c)	(d)
			

TASK 2

Can the following be expressed as a sum of distinct unit fractions?

$$\text{(i) } \frac{3}{4} \quad \text{(ii) } \frac{2}{3} \quad \text{(iii) } \frac{5}{6} \quad \text{(iv) } \frac{2}{7}$$

NOTE: $\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$ does not count because they are not distinct – for some reason Egyptians did not want to write the same fraction symbol more than once!

Are your expressions unique?

(v)Challenge yourself with any other fraction.

Can you come up with a method that works every time (better than trial and error)? Is it always better than trial and error?

EXTENSION: Can you prove that there is an infinite number of ways to express any given fraction as a sum of unit fractions?

TASK 3: The **Greedy Algorithm**

Fibonacci worked on this problem for a long time and came up with many different approaches, one of which is the “greedy algorithm”.

The greedy algorithm takes any proper fraction (< 1) and applies:

- Take away the largest possible unit fraction
- Repeat until you are left with a unit fraction

(i) Try this with some of the previous fractions.

(ii) Try with $\frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$.

Is it always the most efficient? Can you find nicer expressions for these fractions? (you can sometime use previous results.

(iii) **Programme this algorithm** on Python

EXTENSION: Fibonnacci got stuck with $\frac{5}{121}$. Try this in your programme.

Can you find a nicer expression than with the greedy algorithm?

How about $\frac{31}{311}$?

TASK 4: (extension)

Many people have worked on these problems over the year and here are some of the formulas found that give nicer results than the greedy algorithm often does

Try with some values of *p* and *q* (and *n*), or prove:

$$\frac{2}{pq} = \frac{1}{\frac{1}{2}p(p+q)} + \frac{1}{\frac{1}{2}q(p+q)}$$

$$\frac{n}{pq} = \frac{1}{\frac{1}{n}p(p+q)} + \frac{1}{\frac{1}{n}q(p+q)}$$

$$\frac{2}{p} = \frac{1}{p} + \frac{1}{2p} + \frac{1}{3p} + \frac{1}{6p}$$

Fibonacci came up with

$$\frac{p}{pq-1} = \frac{1}{q} + \frac{1}{q(pq-1)}$$

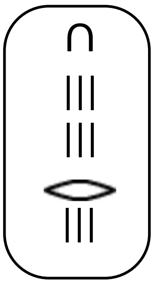
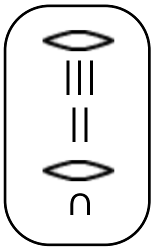
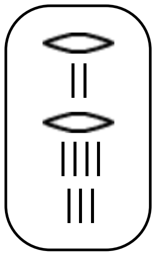
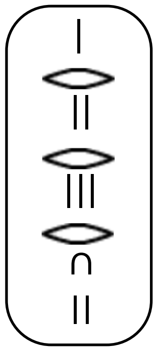
EXTENSION: The Erdös-Strauss conjecture is that you can always express

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

This has been proven for all $n < 10^{17}$

ANSWERS:

<u>TASK 1:</u>

(a)	(b)	(c)	(d)
			
$10+6+\frac{1}{3}=\frac{49}{3}$	$\frac{1}{5}+\frac{1}{10}=\frac{3}{10}$	$\frac{1}{2}+\frac{1}{7}=\frac{9}{14}$	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{12}=\frac{23}{12}$

<u>TASK 2:</u>

$$\text{(i)}\quad \frac{3}{4}=\frac{1}{2}+\frac{1}{4}$$

$$\text{(ii)}\quad \frac{2}{3}=\frac{1}{2}+\frac{1}{6}$$

$$\text{(iii)}\quad \frac{5}{6}=\frac{1}{2}+\frac{1}{3}$$

$$\begin{aligned}\text{(iv)}\quad \frac{2}{7}&=\frac{1}{5}+\frac{1}{12}+\frac{1}{420}\\ &=\frac{1}{7}+\frac{1}{8}+\frac{1}{56}\\ &=\frac{1}{6}+\frac{1}{14}+\frac{1}{21}\end{aligned}$$

(v) verify – propose an alternate expression using extension, put on path of extension

EXTENSION: Consider that unit fractions can be expressed as sums of unit fractions
i.e. $\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$ and $\frac{1}{3}=\frac{1}{4}+\frac{1}{12}$ and $\frac{1}{4}=\frac{1}{5}+\frac{1}{20}$ ….

consider $\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}$

Means that any expression in terms of unit fractions can be broken down infinitely into more unit fractions.

TASK 3:

(ii)

$\frac{3}{7}$:	$\frac{3}{7}-\frac{1}{2}<0$ too big $$\frac{3}{7}-\frac{1}{3}=\frac{2}{21}$$ $\frac{2}{21}-\frac{1}{10}<0$ too big $$\frac{2}{21}-\frac{1}{11}=\frac{1}{231}$$ $$\frac{3}{7}=\frac{1}{3}+\frac{1}{11}+\frac{1}{231}$$ (and yet $\frac{3}{7}=\frac{1}{4}+\frac{1}{7}+\frac{1}{28}$ – not algorithmic, but nicer)
$\frac{4}{7}$:	$$\frac{4}{7}=\frac{1}{2}+\frac{1}{14}$$
$\frac{5}{7}$:	$$\frac{5}{7}=\frac{1}{2}+\frac{1}{5}+\frac{1}{70}$$ (and yet $\frac{5}{7}=\frac{1}{2}+\frac{1}{7}+\frac{1}{14}$ is a little nicer)
$\frac{6}{7}$:	$$\frac{6}{7}=\frac{1}{2}+\frac{1}{3}+\frac{1}{42}$$

EXT:

$$\frac{5}{121}=\frac{1}{33}+\frac{1}{121}+\frac{1}{363}$$

Vs

$$\frac{5}{121}=\frac{1}{25}+\frac{1}{757}+\frac{1}{763309}+\cdots$$

31/311 = 1/12 + 1/63 + 1/2799 + 1/8708.