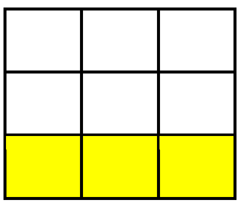
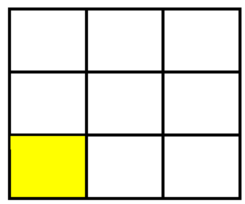
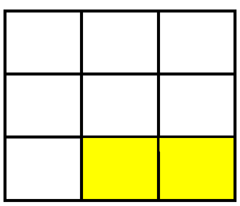
Light Chasing

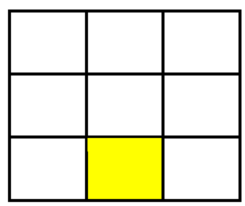
"Light chasing" is a method similar to Gaussian elimination which always solves the puzzle (if a solution exists), although with the possibility of many redundant steps. In this approach, rows are manipulated one at a time starting with the top row. All the lights are disabled in the row by toggling the adjacent lights in the row directly below. The same method is then used on the consecutive rows up to the last one. The last row is solved separately, depending on its active lights.

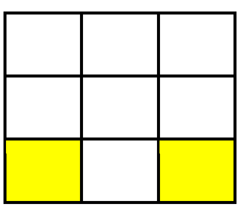
These are all the possibilities (effectively) for the last row. Can you solve them all, and hence solve any 3x3 grid?



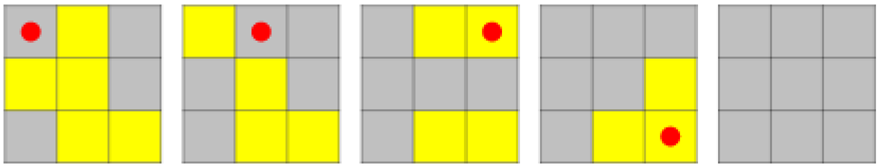








Lights Out Puzzle



A one-person game played on a rectangular lattice of lamps which can be turned on and off. A move consists of flipping a "switch" inside one of the squares, thereby toggling the on/off state of this and all four vertically and horizontally adjacent squares. Starting from a randomly chosen light pattern, the aim is to turn all the lamps off. The problem of determining if it is possible to start from set of all lights being on to all lights being off is known as the "all-ones problem." As shown by Sutner (1989), this is always possible for a square lattice (Rangel-Mondragon).

This can be translated into the following algebraic problem.

1. Each lamp configuration can be viewed as a matrix L with entries in $\mathbb{Z}_2$ (i.e., a (0,1)-matrix, where each 1 represents a burning light and 0 represents a light turned off. For example, for the 3×3 case,

$$\mathsf{L} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad (1)$$

2. The action of the switch placed at (i, j) can be interpreted as the matrix addition $\mathsf{L} + \mathsf{A}_{i\,j}$, where $\mathsf{A}_{i\,j}$ is the matrix in which the only entries equal to 1 are those placed at (i, j) and in the adjacent positions; there are essentially three different types of matrices $\mathsf{A}_{i\,j}$, depending on whether (i, j) is a corner entry,

$$\mathsf{A}_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2)$$

a side entry,

$$\mathsf{A}_{12} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or a middle entry,

$$\mathsf{A}_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

3. Since matrix addition is commutative, it follows that the order in which the moves are performed is irrelevant.

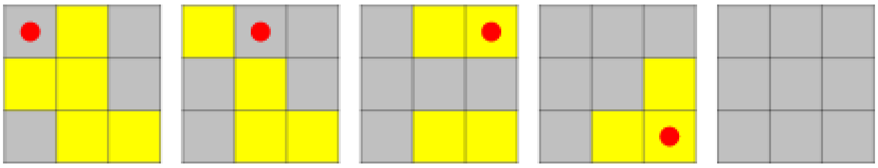
4. Every winning combination of moves can be expressed mathematically in the form:

$$\mathsf{L} + \sum_{i,j} x_{i\,j}\,\mathsf{A}_{i\,j} = \mathsf{0}.$$

Here, $\mathsf{0}$ denotes the zero matrix, which corresponds to the situation where all lights are turned off, and each coefficient $x_{i\,j}$ represents the number of times that switch (i, j) has to be pressed. Because we are solving the equations (mod 2), they can therefore be written in the equivalent form

$$\sum_{i,j} x_{i\,j}\,\mathsf{A}_{i\,j} = \mathsf{L}. \qquad (6)$$

Furthermore, it suffices to consider 0 and 1 as the only possible values for $x_{i\,j}$. Hence, the above equality is in fact a system of linear equations in the indeterminates $x_{i\,j}$ over the field $\mathbb{Z}_2$.



$L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ represents the yellow squares at the start. Aiming to change this to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by adding the representations of switching the red lights.

Complete the matrices A_{ij} for what the red lights do above:

$$A_{11} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A \quad = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \quad A \quad = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Complete the additions to check the solution:

$$\text{L} + A_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + A_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + A \quad = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + A \quad = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} + \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

You've just checked the solution; to solve it requires solving this matrix equation:

For example, the system corresponding to the initial (left) light pattern above can be written as

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}. \qquad (7)$$

It has exactly one solution: $((1, 1, 1), (0, 0, 0), (0, 0, 1))$, which means that the game is solved by pressing the switches $(1, 1)$, $(1, 2)$, $(1, 3)$, and $(3, 3)$ (corresponding to the red dots in the figure above). Since the matrix of the above system of equations has maximal rank (it is a 9×9 matrix with nonzero determinant), the game on a 3×3-lattice is always solvable.