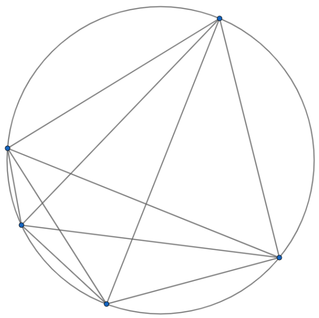
CHALLENGE:

A LONG ANSWER FOR A SHORT QUESTION

(a) Drawing the circle, we should be able to count the number of regions created by considering new post. But be careful: they do not have to be evenly distributed. (in fact, if they are, you will potentially lose a region!

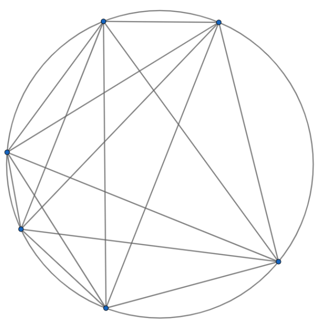


Just by carefully counting, we should be able to determine 16 regions

Now the pattern
1, 2, 4, 8, 16 should start setting off some alarms…. But we need to check.

Will the next one be 32?

(b) Carefully drawing it (this is where you need to make sure that no three fences intersect in the same point – you always can avoid it!)



And bizarrely, at the max, we can get not 32 but

31 regions…!

1, 2, 4, 8, 16, 31 kind of breaks the pattern.

But can we figure out what the pattern is? Because there obviously is one.

Using the difference method, we can maybe start seeing a pattern lying quite deep.

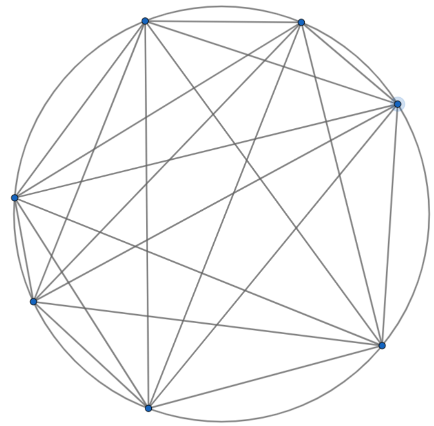
1		**2**		**4**		**8**		**16**		**31**
	1		2		4		8		15	
		1		2		4		7		
			1		2		3			
				1		1				

4 layers deep we get a constant difference! If we continue the pattern, can we predict the next one?

1		2		4		8		16		31		**57**
	1		2		4		8		15		**26**	
		1		2		4		7		**11**		
			1		2		3		**4**			
				1		1		**1**				

Is the next one 57?

(c) We can draw and count…(have fun)



… it is!

1 , 2 , 4 , 8 , 16 , 31 , 57 , …

THIS IS WHERE I EXPECTED YOU TO STOP

Thinking in terms of an n^{th} term formula, every time we go one layer deeper with the difference method, we are dealing with one power higher, or one extra way that *n* interacts with *n*. Turns out I overreached (sorry) and it's actually not possible for you to do this at this stage in maths without knowing some formulas we look at considerably later.

We can still use arithmetic sequences & series to find the n^{th} term formula for the next layer up:

- The penultimate layer is

1 , 2 , 3 , 4 … therefore given by $u_n = 1 + (n-1) \times 1 = n$

$$\boxed{1 , 2 , 3 , 4 \ldots \ : \ \boldsymbol{u_n = n}}$$

- The next level is a little more difficult to model : 1 , 2 , 4 , 7 , 11 , …

We would like the same logic $v_n = 1 + (n-1)d$ where *d* is the gap found with u_{n-1} (previous sequence, 1 position earlier than *v*). But we can't just multiply it by the place value. What is happening is that every time we go up a term, we are going up by the sum of u_{n-1} terms at that point.

$$v_n = 1 + S_{n-1}$$

Where

$$S_{n-1} = \frac{n-1}{2}(2 \times 1 + (n-1-1) \times 1) = \frac{(n-1)n}{2} = \frac{n^2 - n}{2}$$

So

$$v_n = 1 + \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n + 2)$$

$$\boxed{1 , 2 , 4 , 7 , 11 \ldots : \ \boldsymbol{v_n = \frac{1}{2}(n^2 - n + 2)}}$$

THIS IS AS FAR AS YOU CAN GO WITH GCSE TOOLS

THIS IS WHERE THE CRAZY STUFF STARTS

For anyone interested, to continue we have to repeat this logic, but as soon as we look at sums with squares and cubes, it's not longer "arithmetic" (although you could employ the same logic as we have just done to find said formulas, given that the difference between square numbers is an arithmetic sequence! – as seen in the extension in class)

I went on a bit of a deep dive trying to prove this one (after I gave it to you… oops), but I had to use these properties, which we have not seen yet:

$$1 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$1 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

I won't go into the detail, but you can have a look yourself if you're so motivated but here are my results:

if:

z_n:	1		2		4		8		16		31		**57**
w_n:		1		2		4		8		15		**26**	
v_n:			1		2		4		7		**11**		
u_n:				1		2		3		**4**			
					1		1		**1**				

We saw

$$v_n = 1 + S_{(u_{n-1})}$$
$$v_n = \frac{1}{2}(n^2 - n + 2)$$

- Similarly, (for $1, 2, 4, 8, 15, 26, \ldots$)

$$w_n = 1 + S_{(v_{n-1})}$$

We can find (using previous formulas)

$$S_{v_n} = \frac{1}{6}n(n^2 + 5)$$
$$w_n = 1 + \frac{1}{6}(n-1)((n-1)^2 + 5)$$

And

$$\boxed{w_n = \frac{1}{6}(n^3 - 3n^2 + 8n)}$$

(try it with different values of *n*, it really does give 1 , 2 , 4 , 8 , 15 , 26 …)

- Finally, (for $1, 2, 4, 8, 16, 31, 57, \ldots$)

$$z_n = 1 + S_{(w_{n-1})}$$

Expanding and applying formulas, we can find:

$$S_{w_n} = \frac{1}{24}n(n+1)(n^2 - 3n + 14)$$
$$\Rightarrow z_n = 1 + \frac{1}{24}(n-1)n((n-1)^2 - 3(n-1) + 14)$$

And after far too much work:

$$\boxed{z_n = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)}$$

So. I hope you all make successful chicken farmers some day!