Find the remainder

** - Facilitator

Introduction

You could start this activity with just the question "What is the remainder when 3^{2001} is divided by 7?". Students may try to use there calculators – but most calculators won't be able to calculate 3^{2001}. The answer is 954 digits long!! It is:

5 243 613 755 167 954 828 979 923 857 493 981 711 587 187 462 305 565 553 435 664
035 432 058 798 681 826 467 874 459 409 594 043 333 226 826 852 088 435 829 766
314 105 307 626 525 942 543 242 606 280 915 445 618 737 140 269 931 538 144 588
885 434 961 570 557 957 428 044 513 272 178 232 670 643 501 501 943 441 807 456
777 105 716 207 816 452 294 413 870 938 105 653 077 475 718 233 955 931 993 749
571 031 134 308 987 885 939 219 621 006 883 925 988 857 792 735 239 251 412 587
387 422 273 316 333 312 065 906 438 454 426 044 272 265 990 459 807 203 799 275
098 439 667 269 473 952 525 386 702 435 819 190 926 993 671 431 212 971 081 241
209 719 933 273 595 890 540 220 281 137 133 747 311 608 282 671 064 147 324 069
732 417 549 083 887 453 233 235 393 318 550 535 683 885 805 663 809 444 260 250
347 729 366 521 948 221 255 956 122 138 193 777 640 388 129 452 213 424 943 505
054 039 274 170 717 846 277 739 105 933 248 455 589 414 255 829 810 787 449 232
835 472 799 226 232 852 856 734 803 555 466 806 242 315 240 357 878 079 638 470
575 800 169 895 813 826 209 383 776 296 743 889 809 027 737 938 309 469 031 863
126 328 378 066 808 141 124 600 612 569 413 823 932 072 501 849 303 292 870 244
001 701 140 931 218 464 169 473 646 337 349 975 021 457 162 331 320 003

This is a puzzle that shows you can use patterns in smaller numbers to work out the answer to a bigger problem.

The "remainder when divided by 7", is called mod 7 in maths. The mod stands for modulo.

35 mod 7 = 0 because there is no remainder when you divide by 7.

36 mod 7 = 1 because there is a remainder of 1 when you divide by 7.

You could do the following activity to discuss the idea:

Ask students to get into groups of 7. How many people are left over? This is the remainder.

Solution

Power of 3	Equal to *(x)*	Remainder when divided by 7 *x mod 7*
3^0	1	1
3^1	3	3
3^2	9	2
3^3	27	6
3^4	81	4
3^5	243	5
3^6	729	1
3^7	2187	3
3^8	6561	2
3^9	19683	6
3^{10}	59049	4
3^{11}	177147	5

There is a pattern that is repeating itself every six numbers:

Remainder when divided by 7	Powers of 3
1	0, 6, 12,...
3	1, 7, 13,...
2	2, 8, 14,...
6	3, 9, 15,...
4	4, 10, 16, ...
5	5, 11, 17, ...

So we just need to find out which list 2001 would be in.

The first list 0,6,12 are all multiples of 6 (0 mod 6)

The second list 1,7,13 are all one more than a multiple of 6 (1 mod 6)

The third list 2, 8, 14 are all two more than a multiple of 6 (2 mod 6)

etc

Remainder when divided by 7	Powers of 3	Remainder when divided by 6
1	0, 6, 12,...	0
3	1, 7, 13,...	1
2	2, 8, 14,...	2
6	3, 9, 15,...	3
4	4, 10, 16, ...	4
5	5, 11, 17, ...	5

So we just need to find the remainder when we divide 2001 by 6 (i.e. what is 2001 mod 6).

2001 mod 6 = 3, so 2001 would appear in the list 3,9,15, ... so the answer is **6.**

Here is another way to think about this problem (without working out the powers!!) This is the power of mod arithmetic:

Power of 3	Equal to *(x)*	Just multiply previous answer by 3	Remainder when divided by 7 *x mod 7*	
3^0	1	1	1	
3^1	~~3~~	3×1=3	3	
3^2	~~9~~	3×3=9=2	2	
3^3	~~27~~	3×2=6	6	
3^4	~~81~~	3×6=18=4	4	
3^5	~~243~~	3×4=12=5	5	
3^6	~~729~~	3×5=15=1	1	
3^7	~~2187~~	3×1=3	3	
3^8	~~6561~~	3×3=9=2	2	
3^9	~~19683~~	3×2=6	6	
3^{10}	~~59049~~	3×6=18=4	4	
3^{11}	~~177147~~	3×4=12=5	5	

We can then say $3^{2001} = 3^3 \times 3^{1998} (mod 7)$

$$= 3^3 \times (3^6)^{333} (mod 7)$$

$$= 6 \times (1)^{333} (mod 7)$$

$$= 6 \times 1 (mod 7)$$

=6 (mod 7)

Chosen because $3^6 = 1$

So answer is **6**.

Or you can think about $3^{2001} = 3^{201} = 3^{21} = 3^3 = 6\ (mod 7)$

Extension

This problem comes from from the Nrich website https://nrich.maths.org/373.

What is the remainder when 5^{3019} is divided by 7? **Answer is 5.**