Domino tilings

African Maths Initiative

**\*\* – Facilitator**

**Introduction**

The idea of this exercise is to explore the possibility or otherwise of tiling (covering) certain shapes by dominos. While the question appears geometric, there the combinatorics of the situation also turns out to be imporant. One idea that learners should get from this exercise is that sometimes a construction is possible while other times it will not be; good students should also start to be able to make arguments (proofs!) for *why* a certain tiling is not possible. The proof of the most advanced example introduces a basic idea in combinatorial arguments, that or colouring.

Experimentation can be helped by the availability of squared paper, or paper and scissors to make the dominos!

**Solution**

In part a. tilings clearly exists (this could be done at the beginning together so that students get the idea). Experimentation will suggest that in part b. there is *no solution*, and good students will also spot the fact that the reason is that there is an *odd* number of squares. Indeed, removing one square as in part c. will give a shape that can be tiled again. For part d. experimentation quickly gives a tiling; for part e., it can be shown by trying essentially all possibilities that there is *no solution* again. How so, since there is an even number of squares in both cases, and we removed two corners in each case? Well, the trick is to think of a chessboard colouring of our shape (it should be suggested to good students to think of a chessboard, or explain to them what that looks like, but initially without further comment). The point is that no matter how one puts down a domino, it will always cover *one* black and *one* white square. So in part e. no tiling can exist since the coloured version has a different number of black and white squares:

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This argument will also show that no tiling in case f. exists either; this case would be impossible to do by trial and error.

**Extension**

Students can go on to discuss which shapes can be tiled, and which cannot. For example,

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clearly cannot be tiled; this is very easy to see directly, but the colouring argument also proves it. A more interesting question is whether there are any shapes that cannot be tiled, even though they contain the same number of black and white squares. This could be set as a challenge to a good class. A good student might find the following example:

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(placing this “shape” on a chessboard so that one square becomes white, the other black). While in some sense this is “cheating”, this would be a good point to talk about *connectedness* of our shapes. A more interesting example, “doubling up” the previous example with 4 squares, is

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While this shape has 4 black and 4 white squares, it still cannot be tiled, as a simple experimentation shows. Further references:

https://en.wikipedia.org/wiki/Mutilated\_chessboard\_problem

https://en.wikipedia.org/wiki/Domino\_tiling

https://arxiv.org/pdf/math/0501170.pdf