Mastermind

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Abstract

The commercial game called "Mastermind" can be used to teach a number of mathematical and logical concepts. This paper discusses the game and various strategies not only to play the game, but to teach some mathematics using the game as an example. Invicta Plastics was the original producer of the game in 1971 and it holds the worldwide Trademark for the name. Here we study the game with 4 slots and 6 colors, although other versions are obviously possible.

1 The Game

Mastermind is a game played between two people. One selects a secret code and the other makes successive guesses to determine the secret code by making guesses for each of which the first person gives an indication of how close the guess is.

The code is a sequence of four colors chosen from a set of six as is each guess. In the physical game, there are colored pegs that fit into four adjacent holes. The colors may repeat in the sequence, so altogether, there are $6^4 = 1296$ different possible codes (and guesses). The colors vary in different versions of the game, so we will arbitrarily use these six: red, green, blue, yellow, cyan and purple. In this paper, if we just want to talk about a general sort of position, we may just use the numbers from 1 to 6 to represent the colors. Thus 1123 might represent "red red yellow blue" or "green green yellow purple".

When a guess is made, codemaker compares the colors in the guess with the colors in the secret code. For every position that has the correct color in the correct position, he places a small dark peg (which we'll call "black" in what follows). After all those are accounted for, he places a small white peg for any other colors that are correct, but in the wrong position. The game continues until the guesser matches the secret code exactly.

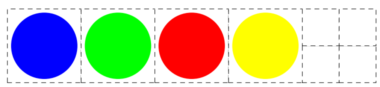
In the example in the photo above, the secret code is green-cyan-red-purple and is hidden in a compartment that only the codemaker can see. The guesser's first guess in this game was yellow-yellow-cyan-cyan. There are no pegs that are exactly right, but there is a single cyan in the secret code so the codemaker places zero dark pegs and one white peg to the right of the pattern. The next guess, purple-red-red-yellow, has one perfect match (red in the third position) and one correct color in the wrong position (the purple peg). Thus the response is one dark peg (red instead of black, in this version of the physical game) and one white peg. The game continues in this way until the sixth move which is totally correct and therefore receives four small dark pegs.

If you've never played the game before, it's probably useful to verify that the patterns guessed on the other rows correspond to the small peg patterns to the right of each row's guess.

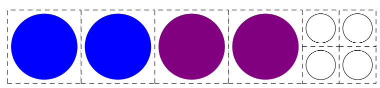
2 Reasoning About Positions

Let's try to learn some strategy by examining initial parts of some games. There are a lot of questions, the answers to which appear in Section 5

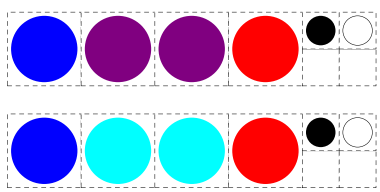
1. In the example below, there are no pegs, so none of the four colors blue, green, red or yellow appear in the final pattern. This means that the final pattern consists of just cyan and purple colors. How many possible patterns are there? What are they? What might be a good second guess?



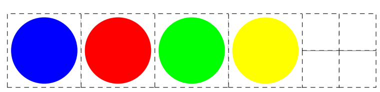
2. The example below of an initial guess (where the response is four white pegs) completely determines the solution. What is it?



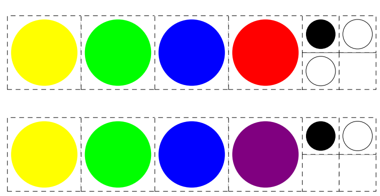
3. In the following game, changing both the purple pegs to cyan pegs makes no change in the results. Can we therefore conclude that there are no purples or cyans in the final pattern?

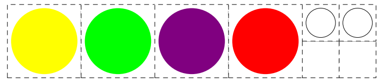


4. Here's the next step in the game above. Now what can you conclude?

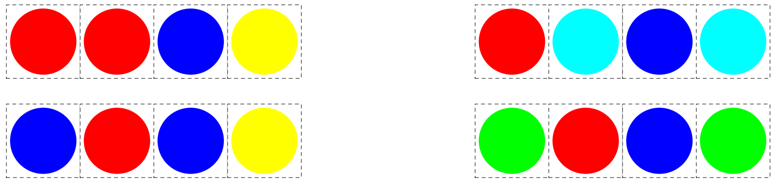


5. What can you conclude about the secret pattern if the following represent the first three guesses in a game?

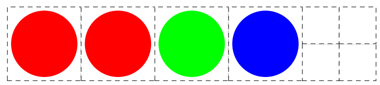




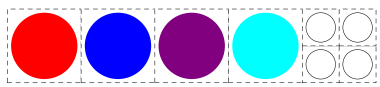
6. Assuming that you are playing the game above, which of the following are possible secret positions?



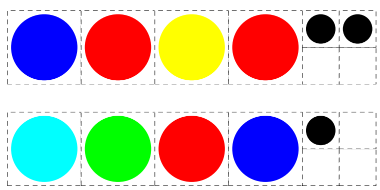
7. What can be concluded from the following initial move? It seems like we didn't learn anything: the response of no pegs means that there were zero matches.



8. Given this result (4 white pegs) for the initial guess, how many possible secret values are there? What are they?



9. (*) Given the following first two moves, there are only five possibilities for the secret position. What are they?



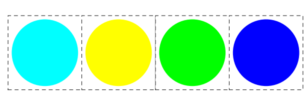
10. (*) Find all 16 possible patterns, given the initial information from the game started in question 5 above.

3 Equivalent Strategies

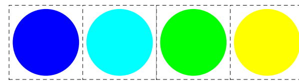
If you have a good strategy to solve Mastermind games and you use it over and over again, pretty soon your opponent will realize it and will choose a pattern that is the last to be discovered by your strategy. In order to avoid this, what you need to do is select a different sequence of moves each time, although they may be equivalent to your good strategy.

3.1 Rearrangement (permutation) of the peg positions

If the pegs are numbered $1, 2, 3, 4$ there's no reason that you can't mentally think of them in a different order, say $4, 1, 3, 2$ and you modify your strategy by putting your guesses on the renumbered pegs. Using the rearrangement above as an example, suppose your good strategy begins by guessing the following pattern:



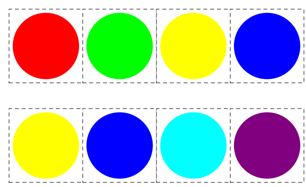
Then your rearranged (permuted) strategy would begin with the following guess:



3.2 Color permutation

In a similar manner as with the peg permutations, you can also mentally rearrange the colors as well. In fact, it might be best to think of your strategy as a sequence of numbers from 1 to 6, so you might have a strategy that looks like this: "My first guess is 1234, then, if the response is BB, my next guess will be 3456, et cetera. When you play an actual game, you simply assign the colors at random to the numbers and play the strategy.

Using the example strategy above, if your assignment were: $1 \leftrightarrow R$, $2 \leftrightarrow G$, $3 \leftrightarrow Y$, $4 \leftrightarrow B$, $5 \leftrightarrow C$, and $6 \leftrightarrow P$, then the two guesses in the strategy described in the previous paragraph would look like:



There is, of course, no reason not to mix both kinds of permutations. Here are four essentially-equivalent first guesses:

All have two colors the same, but in different positions.

4 The Best Guess

A good guess should be able to divide the collection of possible secret patterns into piles that are as equal in size as possible. There are 14 possible responses to any guess (indicated by the numbers or black and white pegs):

$$[BBBB] \quad [BBWW] \quad [BWWW] \quad [WWWW]$$
$$[BBB] \quad [BBW] \quad [BWW] \quad [WWW]$$
$$[BB] \quad [BW] \quad [WW]$$
$$[B] \quad [W]$$
$$[\quad]$$

Exercise: Why is $BBBW$ missing from this list?

Imagine, for every possible guess, going through the list of remaining possible secret patterns and finding out which patterns fall into each of the 14 categories above. The guess that most evenly divides up the results is, in some sense, the best.

Without going into details, in fact a very reasonable method to measure just how evenly-divided the results are is to measure the "entropy" (in the information science sense) of the distribution. If the probabilities of falling into each of the fifteen categories above are $p_1, p_2, \ldots, p_{14}$, then the entropy H associated with that distribution is given by:

$$H = -(p_1 \log p_1 + p_2 \log p_2 + \cdots + p_{14} \log p_{14}),$$

where any term with $p_i = 0$ is assumed to be zero in spite of the fact that the logarithm of zero is undefined. For technical reasons, the logarithms are usually chosen to be base-2, but for our purposes it doesn't really matter, since if we use a different base, all the logarithms are just multiplied by the same constant value. In this article, however, we will always compute the entropy using base-2 logarithms.

The more evenly-divided a distribution, the larger will be the value of the entropy.

What is a good first guess? There are 5 fundamentally different initial moves: $AAAA$, $AAAB$, $AABB$, $AABC$ and $ABCD$, where A, B, C and D represent different colors. We also ignore the order, so we'll consider $AABC$, $ABCA$, $CABA$, and so on, to the essentially the same. It's not too hard to figure out how many of the secret starting positions would yield each of the 14 possible responses given those initial moves. Here's the chart:

	AAAA	AAAB	AABB	AABC	ABCD
[BBBB]	1	1	1	1	1
[BBWW]	0	3	4	5	6
[BWWW]	0	0	0	4	8
[WWWW]	0	0	1	2	9
[BBB]	20	20	20	20	20
[BBW]	0	24	32	40	48
[BWW]	0	27	36	84	132
[WWW]	0	0	16	44	136
[BB]	150	123	114	105	96
[BW]	0	156	208	230	252
[WW]	0	61	96	222	312
[B]	500	317	256	182	108
[W]	0	308	256	276	152
[ ]	625	256	256	81	16
H	1.49844	2.69343	2.8851	3.0437	3.05667

The numbers in the bottommost lines are the entropies expected for each choice of an initial move type. If you want to obtain the smallest expected number of guesses to solve a puzzle, you probably should choose a starting pattern with all four colors different. It turns out that with this strategy (and careful playing afterward), that you can almost always find the secret code in 5 moves or fewer, but there are a couple of cases where you will be required to make 6 guesses.

If you start with the strategy $AABB$, however, you can always find the secret pattern in 5 or fewer moves, although the average number of moves you'll need to make is a bit larger. There is some evidence for that in the table above since by choosing $AABB$, the largest collection of possibilities you'll have after the first move is 256: the smallest of any of the possible initial moves.

It should be clear, however, that $AAAA$ is a pretty bad first move.

Next, let's consider the game from question 5 in Section 2.

The following tables list, for every possible remaining secret value, the results of guessing all of the possible secret values. For example, in the column under $RRBY$ shows the B/W results if $RRBY$ is the guess used against every possible secret pattern. It's not a very good guess, since there are 7 responses of BB, 6 of BBB, 2 of BBW and one of $BBBB$.

The bottom entry in each column is the value of the entropy, H, for that guess against all possible secret values. There are a bunch of guesses for which the entropy is largest: 2.82782, so at first glance it seems that one of those might be a good candidate for a guess.

	RRBY	RRBG	RBBY	RBBG	RCBY	RCBG	RYBB	RYBC
RRBY	BBBB	BBB	BBB	BB	BBB	BB	BBW	BBW
RRBG	BBB	BBBB	BB	BBB	BB	BBB	BB	BB
RBBY	BBB	BB	BBBB	BBB	BBB	BB	BBWW	BBW
RBBG	BB	BBB	BBB	BBBB	BB	BBB	BBW	BB
RCBY	BBB	BB	BBB	BB	BBBB	BBB	BBW	BBWW
RCBG	BB	BBB	BB	BBB	BBB	BBBB	BB	BBW
RYBB	BBW	BB	BBWW	BBW	BBW	BB	BBBB	BBB
RYBC	BBW	BB	BBW	BB	BBWW	BBW	BBB	BBBB
RYBY	BBB	BB	BBB	BB	BBB	BB	BBB	BBB
BRBY	BBB	BB	BBWW	BWW	BBW	BW	BWWW	BWW
BRBG	BB	BBB	BWW	BBWW	BW	BBW	BWW	BW
CRBY	BBB	BB	BBW	BW	BBWW	BWW	BWW	BWWW
CRBG	BB	BBB	BW	BBW	BWW	BBWW	BW	BWW
GRBB	BB	BBW	BWW	BWWW	BW	BWW	BBW	BW
GRBC	BB	BBW	BW	BWW	BWW	BWWW	BW	BBW
GRBG	BB	BBB	BW	BBW	BW	BBW	BW	BW
H	1.67742	1.67742	2.70282	2.78064	2.70282	2.78064	2.82782	2.82782

	RYBY	BRBY	BRBG	CRBY	CRBG	GRBB	GRBC	GRBG
RRBY	BBB	BBB	BB	BBB	BB	BB	BB	BB
RRBG	BB	BB	BBB	BB	BBB	BBW	BBW	BBB
RBBY	BBB	BBWW	BWW	BBW	BW	BWW	BW	BW
RBBG	BB	BWW	BBWW	BW	BBW	BWWW	BWW	BBW
RCBY	BBB	BBW	BW	BBWW	BWW	BW	BWW	BW
RCBG	BB	BW	BBW	BWW	BBWW	BWW	BWWW	BBW
RYBB	BBB	BWWW	BWW	BWW	BW	BBW	BW	BW
RYBC	BBB	BWW	BW	BWWW	BWW	BW	BBW	BW
RYBY	BBBB	BBW	BW	BBW	BW	BW	BW	BW
BRBY	BBW	BBBB	BBB	BBB	BB	BBW	BB	BB
BRBG	BW	BBB	BBBB	BB	BBB	BBWW	BBW	BBB
CRBY	BBW	BBB	BB	BBBB	BBB	BB	BBW	BB
CRBG	BW	BB	BBB	BBB	BBBB	BBW	BBWW	BBB
GRBB	BW	BBW	BBWW	BB	BBW	BBBB	BBB	BBB
GRBC	BW	BB	BBW	BBW	BBWW	BBB	BBBB	BBB
GRBG	BW	BB	BBB	BB	BBB	BBB	BBB	BBBB
H	2.12661	2.78064	2.70282	2.78064	2.70282	2.82782	2.82782	2.12661

It appears that there are four equally-valuable guesses at this point: all the guesses that yield the entropy value of 2.82782; namely, $RYBB$, $RYBC$, $GRBB$ and $GRBC$.

What makes this example somewhat surprising, however, is that possibly a better choice would be $RBCG$ which is not one of the possible secret patterns. Here are the values it generates in the same order as in the tables above:

	RBCG
RRBY	BW
RRBG	BBW
RBBY	BB
RBBG	BBB
RCBY	BWW
RCBG	BBWW
RYBB	BW
RYBC	BWW
RYBY	BW
BRBY	WW
BRBG	BWW
CRBY	WWW
CRBG	BWWW
GRBB	WWW
GRBC	WWWW
GRBG	WWW
H	3.10846

There are 3 of BWW, 3 of BW, 3 of WWW, and only 1 of each of the other seven: BBW, BB, BBB, WW, $BWWW$, $BBWW$ and $WWWW$. and this yields a value of $H = 3.10846$: quite a bit better.

Exercise: Check a few values of the calculated entropies above. (Remember that $\log_2 x = \log_{10} x / \log_{10} 2$, and there's nothing special about the base-10 logarithm on the right: they can be of any base.)

5 Solutions to "Reasoning" Questions

1. Using C and P to represent Cyan and Purple, respectively, here are the possibilities where each row lists the possibilities for a fixed number of C's:

   $CCCC$
   $CCCP \quad CCPC \quad CPCC \quad PCCC$
   $CCPP \quad CPCP \quad CPPC \quad PCCP \quad PCPC \quad PPCC$
   $CPPP \quad PCPP \quad PPCP \quad PPPC$
   $PPPP$

   Altogether, there are $16 = 2^4$ possibilities, since there are four positions and each can be filled with either a C or a P.

   We'll examine later what makes a good move, but $CCPP$ is not bad.

2. Since there are four white pegs, we have exactly the correct set of colors, but none are in the correct position. The only way this can occur is to reverse the colors, yielding $PPBB$.

3. At first that reasoning seems correct, but see the next item. (It certainly *could* be the case that there are no cyans or purples, but that is not necessarily true.)

4. In fact, the final position could be $CPCP$ or $PCPC$. Check that both of those patterns satisfy the results in the previous item.

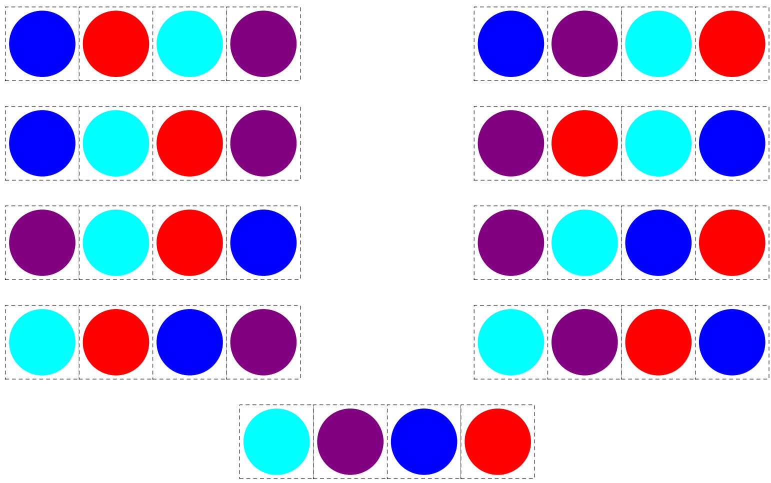
5. The first guess shows that three of the four colors are represented by yellow, green, blue and red. The second guess replaces exactly one of the colors (purple replaces red) and this reduces the number of white pegs, indicating that one of the colors in the secret pattern is red (but not in that position) and

that purple is not one of the other colors. The third guess is the same as the first one, but this time with purple (known not to be in the secret pattern) replacing blue. This time a black peg disappears, so blue is also in the secret pattern, and it is in its correct position.

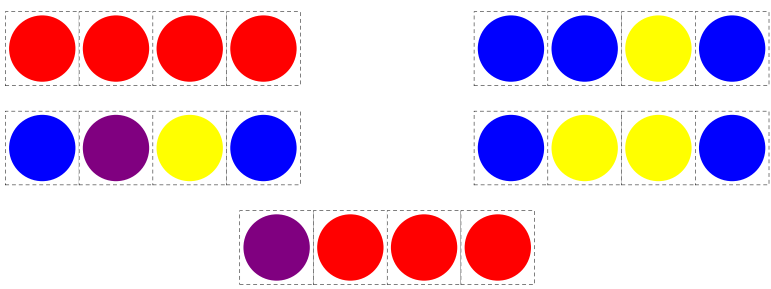
6. Only the second pattern is impossible. To check to see whether a pattern is possible or not, compare all three guesses against the pattern to make sure that all of the small peg results are the same.

7. We actually learn a lot from this sort of negative information. If anything was correct, we'd have a peg, so we know that the entire secret pattern is composed only of cyan, purple and yellow pegs. So in fact, we've reduced the set of possible solutions from $1296 = 6^4$ to $81 = 3^4$ (since there are three possibilities for each of three spots.

8. There are only 9 possibilities. Here they are:



9. Here are the five possible positions:



10. Here is the complete list:

$RRBY$	$RRBG$	$RBBY$	$RBBG$
$RCBY$	$RCBG$	$RYBB$	$RYBC$
$RYBY$	$BRBY$	$BRBG$	$CRBY$
$CRBG$	$GRBB$	$GRBC$	$GRBG$

See Section 4 for a much more interesting analysis of this same game.

6 Other Versions

There is nothing magic about four pegs and six colors; a similar analysis can be made about different-sized games. One of the largest that was every produced had room for five pegs and eight different colors, yielding $8^5 = 32768$ possible positions.

There are, of course, more possible first moves, and for this situation, here are the entropies associated with each of the different possible starting moves:

Pattern	Entropy
11234	2.24462
12345	2.23994
11223	2.20435
11123	2.12302
11122	1.99410
11112	1.83132
11111	1.01704

7 Average Game Length

If we assume the strategy where we always select a guess that maximizes the entropy, what is the distribution of game lengths, and what is the average game length? Here are the results of computer simulations.

A game type of (4,6) indicates that there are 4 pegs and 6 colors, et cetera. The entry in the "First" column is the first move (whether or not it's the "best" first move, entropy-wise). All other moves are selected using the largest entropy value. If there is an asterisk after the first move, it does not have the largest first-move entropy. The numbers in the following columns are the number of games that require that many moves to win. Finally, in the last column is the average game length:

Game	First	1	2	3	4	5	6	7	Average
(4,6)	1234	1	4	71	612	596	12	0	4.41512
(4,6)	1123*	1	10	56	659	566	4	0	4.38194
(4,6)	1122*	1	8	53	609	624	1	0	4.42747
(4,7)	1234	1	5	67	657	1487	184	0	4.73928
(4,7)	1123*	1	10	58	622	1569	141	0	4.73719
(4,7)	1122*	1	8	50	544	1634	164	0	4.78842
(4,8)	1234	1	5	70	596	2468	952	4	5.05005
(4,8)	1123*	1	10	58	512	2639	875	1	5.05249
(4,8)	1122*	1	8	43	416	2429	1187	12	5.16626
(5,6)	11223	1	11	118	2024	5121	499	2	4.76929
(5,6)	11234*	1	8	104	1949	5110	603	1	4.79681
(5,6)	12345*	1	6	92	1664	5229	784	0	4.86034
(5,7)	11234	1	9	121	1796	10255	4587	38	5.15434

It is a bit difficult to run complete analyses of the (5,8) game using the best-guess strategy, but it is easy to test other strategies. For example, here are some distributions obtained by running a "random guess" strategy, where at each point, a possible pattern is chosen as a guess until success is achieved. In every case, 10 complete passes are made through all possible secret patterns. In the table above, the numbers are exact, so the averages can be compared to see how much is lost by not attempting to optimize the strategy somewhat.

Game	1	2	3	4	5	6	7	8	9	10	11	Average
(4,6)	14	109	939	4219	5824	1740	113	2	0	0	0	4.65216
(4,7)	9	129	1060	5047	10416	6214	1072	62	1	0	0	5.03990
(4,8)	11	109	1050	5630	14538	14147	4820	634	21	0	0	5.43145
(4,9)	10	151	1020	5795	17966	24613	12514	3134	391	16	0	5.79803
(5,6)	12	189	2147	14867	37548	20572	2367	56	2	0	0	5.07338
(5,7)	14	200	2346	17536	64179	66137	16567	1072	19	0	0	5.47408
(5,8)	10	158	2330	19045	86469	143494	66384	9329	456	5	0	5.86023
(5,9)	7	175	2344	19940	103648	237739	177808	44264	4384	177	4	6.21833

8 References

1. You can obtain a copy of the code (written in pretty basic C) that I used to generate most of the numbers in this document here:

   `http://www.geometer.org/puzzles/mind.c`

2. `http://mathworld.wolfram.com/Mastermind.html`

3. `http://www.ncbi.nlm.nih.gov/pmc/articles/PMC3022521/`

4. `http://colorcode.laebisch.com/links/Donald.E.Knuth.pdf`

5. `http://en.wikipedia.org/wiki/Mastermind_(board_game)`